Lecture 8 : Differentiation Formulas

As we did with limits and continuity, we will introduce several properties of the derivative and use them along with the derivatives of some basic functions to make calculation of derivatives easier.

Constant Functions and Power Functions

1. **Derivative of a Constatnt**: $\frac{d}{dx}(c) = 0$, if c is a constant.

2. **Power Rule** : If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Example If $g(x) = 2, \quad f(x) = x^3$, find $f'(x)$ and $g'(x)$.

Just as with limits, we have the following rules:

3. **Constant Multiple Rule** : $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$, where c is a constant and f is a differentiable function.

Proof Suppose $f(x)$ is a differentiable function and $g(x) = cf(x)$ for some constant c. Then

$$\frac{d}{dx}[cf(x)] = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0} c\frac{f(x+h)-f(x)}{h}$$

$$= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{d}{dx}f(x).$$

Example Let $f(x) = x^3$, find $f'(x), \quad f''(x), \quad f^{(3)}(x)$ and $f^{(4)}(x)$.

4. **The Sum Rule** if f and g are both differentiable at x, then $f+g$ is differentiable at x and

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof

$$\frac{d}{dx}[f(x)+g(x)] = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} == \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

In a similar manner we get the difference rule:

5. **The Difference Rule** if f and g are both differentiable at x, then $f-g$ is differentiable at x and

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Example Find the derivative of the function $f(x) = x^2 + 2x + 4$.

Example Find the derivative of the function $f_1(x) = x^{12} - 10x^6 + 3x + 1$.

6. **Product Rule**: If f and g are both differentiable at x, then $f\cdot g$ is differentiable at x and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Proof

$$\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\Big(g(x+h)-g(x)\Big)+\Big(f(x+h)-f(x)\Big)g(x)}{h}$$

$$= \left(\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right)\lim_{h\to 0} f(x+h) + \left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)g(x)$$

$$= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example Let $k(x) = x(x^2+2x+4)$, find $k'(x)$.

Example Let $F(t) = (t^2+4)(2t^3+t^2)$. Find $F'(t)$.

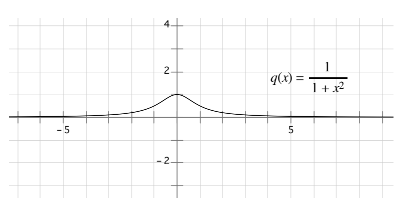
7. Let g be differentiable and non-zero at x, then $\quad \dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = -\dfrac{g'(x)}{(g(x))^2}$

Proof

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0}\frac{g(x)-g(x+h)}{g(x+h)g(x)h} = -\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\lim_{h\to 0}\frac{1}{g(x+h)g(x)}$$

$$= -g'(x)\frac{1}{(g(x))^2}. = -\frac{g'(x)}{(g(x))^2}$$

Example The curve $y = 1/(1+x^2)$ is called a **Witch of Maria Agnesi**. Find the equation of the tangent line to the curve at the point $(-1, 1/2)$.



Now we can combine rules 6 and 7 to get the quotient rule:

8. If f and g are differentiable at x and $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Example Let $K(x) = \dfrac{x^3 + x^2 + 1}{x^4 + 1}$, find $K'(x)$. What is $K'(1)$?

Note we should see if we simplify a function with cancellation before we rush into using the quotient rule.

Example Find the derivative of $L(x) = \dfrac{x^6 + x^4 + x^2}{x^2}$.

General Power Functions

When n is a positive integer $x^n = x \cdot x \cdot x \cdot \cdots \cdot x$, where the product is taken n times. We define $x^0 = 1$ and $x^{-n} = 1/x^n$. We can use the quotient rule to show that

9. If n is a positive integer $\dfrac{d}{dx}(x^{-n}) = -nx^{-n-1}$

Example If $H(x) = 2/x^2 + 3/x^3 + 4/x^4 + 1$ find $H'(x)$.

10. $\boxed{\text{If } k \text{ is any real number } \dfrac{d}{dx}(x^k) = kx^{k-1}}$

Example Find the derivative of $f(x) = \dfrac{\sqrt{x} + x^2 + x^{1/3}}{x^{\sqrt{2}}}$.

Summary of the rules of differentiation

If f and g are differentiable functions, c is a constant and k is any real number, then

$$\tfrac{d}{dx}(c) = 0 \qquad \tfrac{d}{dx}(x^k) = kx^{k-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = gf' + fg' \qquad \left(\tfrac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Old Exam Question

1. For $f(x) = \sqrt[3]{x^5} + \frac{6}{\sqrt[5]{x^3}}$, find $f'(x)$.

(a) $\frac{5\sqrt[3]{x^2}}{3} - \frac{18}{5\sqrt[5]{x^8}}$

(b) $\frac{3\sqrt[3]{x^2}}{5} - \frac{18}{5\sqrt[5]{x^8}}$

(c) $\frac{5\sqrt[3]{x^2}}{3} + \frac{5}{18\sqrt[5]{x^8}}$

(d) $\frac{3\sqrt[3]{x^2}}{5} - \frac{5}{18\sqrt[5]{x^8}}$

(e) $\frac{3\sqrt[3]{x^2}}{5} + \frac{18}{5\sqrt[5]{x^8}}$

2. Find the equation of the tangent line to the curve $y = \frac{x^3}{3} - x^2 + 1$ which is parallel to the line $y + x = 2$.

Old Exam Question, Solution

1. For $f(x) = \sqrt[3]{x^5} + \frac{6}{\sqrt[5]{x^3}}$, find $f'(x)$.

$f(x) = x^{5/3} + 6x^{-3/5}$. Thus

$$f'(x) = \frac{5}{3}x^{2/3} + 6\left(-\frac{3}{5}x^{-8/5}\right) = \frac{5\sqrt[3]{x^2}}{3} - \frac{18}{5\sqrt[5]{x^8}}.$$

2. Find the equation of the tangent line to the curve $y = \frac{x^3}{3} - x^2 + 1$ which is parallel to the line $y + x = 2$.

The line parallel to the line $y + x = 2$ will have the same slope, namely -1. So we need to find the point on the curve which has slope -1. $y' = x^2 - 2x$. We solve for x given $y' = -1$:

$$\begin{aligned} x^2 - 2x &= -1 \\ \implies (x-1)(x-1) &= 0 \\ \implies x &= 1 \end{aligned}$$

Plugging into the equation for the curve we see that $y = 1/3$ at this point. Setting $y = mx + b$, plugging in $(1, 1/3)$ for (x, y) and solving for b we see $b = 4/3$. So the equation of the line we are looking for is $y = -x + 4/3$.